Chapter 7

Finding Time and Longitude by Lunar Distances

In celestial navigation, time and longitude are interdependent. Finding one's longitude at sea or in unknown terrain is impossible without knowing the exact time and vice versa. Therefore, old-time navigators were basically restricted to latitude sailing on long voyages, i. e., they had to sail along a chosen parallel of latitude until they came in sight of the coast. Since there was no reliable estimate of the time of arrival, many ships ran ashore during periods of darkness or bad visibility. Spurred by heavy losses of men and material, scientists tried to solve the longitude problem by using astronomical events as time marks. In principle, such a method is only suitable when the observed time of the event is virtually independent of the observer's geographic position.

Measuring time by the apparent movement of the moon with respect to the background of fixed stars was suggested in the 15th century already (*Regiomontanus*) but proved impracticable since neither reliable ephemerides for the moon nor precise instruments for measuring angles were available at that time.

Around the middle of the 18th century, astronomy and instrument making had finally reached a stage of development that made time measurement by lunar observations possible. Particularly, deriving the time from a so-called **lunar distance**, the angular distance of the moon from a chosen reference body, became a popular method. Although the procedure is rather cumbersome, it became an essential part of celestial navigation and was used far into the 19th century, long after the invention of the mechanical chronometer (*Harrison*, 1736). This was mainly due to the limited availability of reliable chronometers and their exorbitant price. When chronometers became affordable around the middle of the 19th century, lunar distances gradually went out of use. Until 1906, the Nautical Almanac included lunar distance tables showing predicted geocentric angular distances between the moon and selected bodies in 3-hour intervals.* After the tables were dropped, lunar distances fell more or less into oblivion. Not much later, radio time signals became available world-wide, and the longitude problem was solved once and for all. Today, lunar distances are mainly of historical interest. The method is so ingenious, however, that a detailed study is worthwhile.

The basic idea of the lunar distance method is easy to comprehend. Since the moon moves across the celestial sphere at a rate of about 0.5° per hour, the angular distance between the moon, M, and a body in her path, B, varies at a similar rate and rapidly enough to be used to measure the time. The time corresponding with an observed lunar distance can be found by comparison with tabulated values.

Tabulated lunar distances are calculated from the geocentric equatorial coordinates of M and B using the cosine law:

$$\cos D = \sin DEC_M \cdot \sin DEC_B + \cos DEC_M \cdot \cos DEC_B \cdot \cos\left(GHA_M - GHA_B\right)$$

or

$$\cos D = \sin DEC_M \cdot \sin DEC_B + \cos DEC_M \cdot \cos DEC_B \cdot \cos\left[15 \cdot \left(RA_M[h] - RA_B[h]\right)\right]$$

D is the geocentric lunar distance. These formulas can be used to set up one's own table with the aid of the Nautical Almanac or any computer almanac if a lunar distance table is not available.

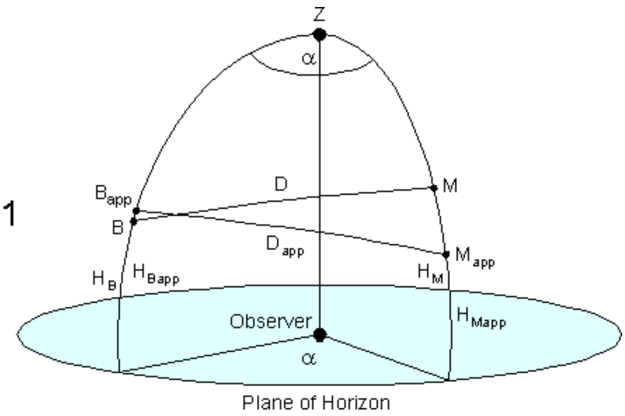
*Almost a century after the original Lunar Distance Tables were dropped, Steven Wepster resumed the tradition. His tables are presently (2004) available through the internet [14].

Clearing the lunar distance

Before a lunar distance measured by the observer can be compared with tabulated values, it has to be reduced to the corresponding geocentric angle by clearing it from the effects of refraction and parallax. This essential process is called **clearing the lunar distance**. Numerous procedures have been developed, among them rigorous and "quick" methods. In the following, we will discuss the almost identical methods by *Dunthorne* (1766) and *Young* (1856). They are rigorous for a spherical model of the earth.

Fig. 7-1 shows the positions of the moon and a reference body in the coordinate system of the horizon. We denote the apparent positions of the centers of the moon and the reference body by M_{app} and B_{app}, respectively. Z is the zenith.

Fig. 7-1



The side D_{app} of the spherical triangle B_{app}-Z-M_{app} is the apparent lunar distance. The altitudes of M_{app} and B_{app} (obtained after applying the corrections for index error, dip, and semidiameter) are H_{Mapp} and H_{Bapp}, respectively. The vertical circles of both bodies form the angle α, the difference between the azimuth of the moon, Az_M, and the azimuth of the reference body, Az_B:

$$\alpha = Az_M - Az_B$$

The position of each body is shifted along its vertical circle by atmospheric refraction and parallax in altitude. After correcting H_{Mapp} and H_{Bapp} for both effects, we obtain the geocentric positions M and B. We denote the altitude of M by H_M and the altitude of B by H_B. H_M is always greater than H_{Mapp} because the parallax of the moon is always greater than refraction. The angle α is neither affected by refraction nor by the parallax in altitude:

$$Az_M = Az_{Mapp} \qquad Az_B = Az_{Bapp}$$

The side D of the spherical triangle B-Z-M is the unknown geocentric lunar distance. If we knew the exact value for α, calculation of D would be very simple (cosine law). Unfortunately, the navigator has no means for measuring α precisely. It is possible, however, to calculate D solely from the five quantities D_{app}, H_{Mapp}, H_M, H_{Bapp}, and H_B.

Applying the cosine formula to the spherical triangle formed by the zenith and the apparent positions, we get:

$$\cos D_{app} = \sin H_{Mapp} \cdot \sin H_{Bapp} + \cos H_{Mapp} \cdot \cos H_{Bapp} \cdot \cos \alpha$$

$$\cos \alpha = \frac{\cos D_{app} - \sin H_{Mapp} \cdot \sin H_{Bapp}}{\cos H_{Mapp} \cdot \cos H_{Bapp}}$$

Repeating the procedure with the spherical triangle formed by the zenith and the geocentric positions, we get:

$$\cos D = \sin H_M \cdot \sin H_B + \cos H_M \cdot \cos H_B \cdot \cos \alpha$$

$$\cos \alpha = \frac{\cos D - \sin H_M \cdot \sin H_B}{\cos H_M \cdot \cos H_B}$$

Since α is constant, we can combine both azimuth formulas:

$$\frac{\cos D - \sin H_M \cdot \sin H_B}{\cos H_M \cdot \cos H_B} = \frac{\cos D_{app} - \sin H_{Mapp} \cdot \sin H_{Bapp}}{\cos H_{Mapp} \cdot \cos H_{Bapp}}$$

Thus, we have eliminated the unknown angle α. Now, we subtract unity from both sides of the equation:

$$\frac{\cos D - \sin H_M \cdot \sin H_B}{\cos H_M \cdot \cos H_B} - 1 \quad = \quad \frac{\cos D_{app} - \sin H_{Mapp} \cdot \sin H_{Bapp}}{\cos H_{Mapp} \cdot \cos H_{Bapp}} - 1$$

$$\frac{\cos D - \sin H_M \cdot \sin H_B}{\cos H_M \cdot \cos H_B} - \frac{\cos H_M \cdot \cos H_B}{\cos H_M \cdot \cos H_B} \quad = \quad \frac{\cos D_{app} - \sin H_{Mapp} \cdot \sin H_{Bapp}}{\cos H_{Mapp} \cdot \cos H_{Bapp}} - \frac{\cos H_{Mapp} \cdot \cos H_{Bapp}}{\cos H_{Mapp} \cdot \cos H_{Bapp}}$$

$$\frac{\cos D - \sin H_M \cdot \sin H_B - \cos H_M \cdot \cos H_B}{\cos H_M \cdot \cos H_B} \quad = \quad \frac{\cos D_{app} - \sin H_{Mapp} \cdot \sin H_{Bapp} - \cos H_{Mapp} \cdot \cos H_{Bapp}}{\cos H_{Mapp} \cdot \cos H_{Bapp}}$$

Using the addition formula for cosines, we have:

$$\frac{\cos D - \cos\left(H_M - H_B\right)}{\cos H_M \cdot \cos H_B} \quad = \quad \frac{\cos D_{app} - \cos\left(H_{Mapp} - H_{Bapp}\right)}{\cos H_{Mapp} \cdot \cos H_{Bapp}}$$

Solving for cos D, we obtain *Dunthorne's* formula for clearing the lunar distance:

$$\cos D = \frac{\cos H_M \cdot \cos H_B}{\cos H_{Mapp} \cdot \cos H_{Bapp}} \cdot \left[\cos D_{app} - \cos\left(H_{Mapp} - H_{Bapp}\right)\right] + \cos\left(H_M - H_B\right)$$

Adding unity to both sides of the equation instead of subtracting it, leads to *Young's* formula:

$$\cos D \;=\; \frac{\cos H_M \cdot \cos H_B}{\cos H_{Mapp} \cdot \cos H_{Bapp}} \cdot \left[\cos D_{app} + \cos\left(H_{Mapp} + H_{Bapp}\right)\right] - \cos\left(H_M + H_B\right)$$

Procedure

Deriving UT from a lunar distance comprises the following steps:

1.

We measure the altitude of the upper or lower limb of the moon, whichever is visible, and note the watch time of the observation, $WT1_{LMapp}$.
We apply the corrections for index error and dip (if necessary) and get the apparent altitude of the limb, $H1_{LMapp}$. We repeat the procedure with the reference body and obtain the watch time $WT1_{Bapp}$ and the altitude $H1_{Bapp}$.

2.

We measure the angular distance between the limb of the moon and the reference body, D_{Lapp}, and note the corresponding watch time, WT_D. The angle D_{Lapp} has to be measured with the greatest possible precision. It is recommended to measure a few D_{Lapp} values and their corresponding WT_D values in rapid succession and calculate the respective average value. When the moon is almost full, it is not quite easy to distinguish the limb of the moon from the terminator (shadow line). In general, the limb has a sharp appearance whereas the terminator is slightly indistinct.

3.

We measure the altitudes of both bodies again, as described above. We denote them by $H2_{LMapp}$ and $H2_{Bapp}$, and note the corresponding watch times of observation, $WT2_{LMapp}$ and $WT2_{Bapp}$.

4.

Since the observations are only a few minutes apart, we can calculate the altitude of the respective body at the moment of the lunar distance observation by linear interpolation:

$$H_{LMapp} = H1_{LMapp} + \left(H2_{LMapp} - H1_{LMapp}\right) \cdot \frac{WT_D - WT1_{LMapp}}{WT2_{LMapp} - WT1_{LMapp}}$$

$$H_{Bapp} = H1_{Bapp} + \left(H2_{Bapp} - H1_{Bapp}\right) \cdot \frac{WT_D - WT1_{Bapp}}{WT2_{Bapp} - WT1_{Bapp}}$$

5.

We correct the altitude of the moon and the angular distance D_{Lapp} for the augmented semidiameter of the moon, SD_{aug}. The latter can be calculated directly from the altitude of the upper or lower limb of the moon:

$$\tan SD_{aug} = \frac{k}{\sqrt{\dfrac{1}{\sin^2 HP_M} - \left(\cos H_{LMapp} \pm k\right)^2} - \sin H_{LMapp}} \qquad k = 0.2725$$

$$(\text{upper limb: } \cos H_{LMapp} - k \qquad \text{lower limb: } \cos H_{LMapp} + k)$$

The altitude correction is:

$$\text{Lower limb: } \quad H_{Mapp} = H_{LMapp} + SD_{aug}$$

$$\text{Upper limb: } \quad H_{Mapp} = H_{LMapp} - SD_{aug}$$

The rules for the lunar distance correction are:

$$\text{Limb of moon towards reference body: } \quad D_{app} = D_{Lapp} + SD_{aug}$$

$$\text{Limb of moon away from reference body: } \quad D_{app} = D_{Lapp} - SD_{aug}$$

The above procedure is an approximation since the augmented semidiameter is a function of the altitude corrected for refraction. Since refraction is a small quantity and since the total augmentation between 0° and 90° altitude is only approx. 0.3', the resulting error is very small and may be ignored.

The sun, when chosen as reference body, requires the same corrections for semidiameter. Since the sun does not show a measurable augmentation, we can use the geocentric semidiameter tabulated in the Nautical Almanac or calculated with a computer program.

6.

We correct both altitudes, H_{Mapp} and H_{Bapp}, for atmospheric refraction, R.

$$R_i\,['] = \frac{p\,[mbar]}{1010} \cdot \frac{283}{T\,[°C] + 273} \cdot \left(\frac{0.97127}{\tan H_i} - \frac{0.00137}{\tan^3 H_i}\right) \qquad i = Mapp, Bapp \qquad H_i > 10°$$

R_i is subtracted from the respective altitude. The refraction formula is only accurate for altitudes above approx. 10°. Lower altitudes should be avoided anyway since refraction may become erratic and since the apparent disk of the moon (and sun) assumes an oval shape caused by an increasing difference in refraction for upper and lower limb. This distortion would affect the semidiameter with respect to the reference body in a complicated way.

7.

We correct the altitudes for the parallax in altitude:

$$\sin P_M = \sin HP_M \cdot \cos\left(H_{Mapp} - R_{Mapp}\right) \qquad \sin P_B = \sin HP_B \cdot \cos\left(H_{Bapp} - R_{Bapp}\right)$$

We apply the altitude corrections as follows:

$$H_M = H_{Mapp} - R_{Mapp} + P_M \qquad H_B = H_{Bapp} - R_{Bapp} + P_B$$

The correction for parallax is not applied to the altitude of a fixed star ($HP_B = 0$).

8.

With D_{app}, H_{Mapp}, H_M, H_{Bapp}, and H_B, we calculate D using *Dunthorne's* or *Young's* formula.

9.

The time corresponding with the geocentric distance D is found by interpolation. Lunar distance tables show D as a function of time, T (UT). If the rate of change of D does not vary too much (less than approx. 0.3' in 3 hours), we can use linear interpolation. However, in order to find T, we have to consider T as a function of D (inverse interpolation).

$$T_D = T_1 + \left(T_2 - T_1\right) \cdot \frac{D - D_1}{D_2 - D_1}$$

T_D is the unknown time corresponding with D. D_1 and D_2 are tabulated lunar distances. T_1 and T_2 are the corresponding time (UT) values ($T_2 = T_1 + 3h$). D is the geocentric lunar distance calculated from D_{app}. D has to be between D_1 and D_2.

If the rate of change of D varies significantly, more accurate results are obtained with methods for non-linear interpolation, for example, with 3-point *Lagrange* interpolation. Choosing three pairs of tabulated values, (T_1, D_1), (T_2, D_2), and (T_3, D_3), T_D is calculated as follows:

$$T_D = T_1 \cdot \frac{\left(D - D_2\right)\cdot\left(D - D_3\right)}{\left(D_1 - D_2\right)\cdot\left(D_1 - D_3\right)} + T_2 \cdot \frac{\left(D - D_1\right)\cdot\left(D - D_3\right)}{\left(D_2 - D_1\right)\cdot\left(D_2 - D_3\right)} + T_3 \cdot \frac{\left(D - D_1\right)\cdot\left(D - D_2\right)}{\left(D_3 - D_1\right)\cdot\left(D_3 - D_2\right)}$$

$$T_2 = T_1 + 3h, \qquad T_3 = T_2 + 3h, \qquad D_1 < D_2 < D_3 \ \ or \ \ D_1 > D_2 > D_3$$

D may have any value between D_1 and D_3.

There must not be a minimum or maximum of D in the time interval [T_1, T_3]. This problem does not occur with a properly chosen body having a suitable rate of change of D. Near a minimum or maximum of D, $\Delta D/\Delta T$ would be very small, and the observation would be erratic anyway.

After finding T_D, we can calculate the watch error, ΔT.

ΔT is the difference between our watch time at the moment of observation, WT_D, and the time found by interpolation, T_D:

$$\Delta T = WT_D - T_D$$

Subtracting the watch error from the watch time, WT, results in UT.

$$UT = WT - \Delta T$$

Improvements

The procedures described so far refer to a spherical earth. In reality, however, the earth has approximately the shape of an ellipsoid flattened at the poles. This leads to small but measurable effects when observing the moon, the body nearest to the earth. First, the parallax in altitude differs slightly from the value calculated for a spherical earth. Second, there is a small parallax in azimuth which would not exist if the earth were a sphere (see chapter 9). If no correction is applied, D may contain an error of up to approx. 0.2'. The following formulas refer to an observer on the surface of the reference ellipsoid (approximately at sea level).

The corrections require knowledge of the observer's latitude, Lat, the true azimuth of the moon, Az_M, and the true azimuth of the reference body, Az_B.

Since the corrections are small, the three values do not need to be very accurate. Errors of a few degrees are tolerable. Instead of the azimuth, the compass bearing of each body, corrected for magnetic declination, may be used.

Parallax in altitude:

This correction is applied to the parallax in altitude and is used to calculate H_M with higher precision before clearing the lunar distance.

$$\Delta P_M \approx f \cdot HP_M \cdot \left[\sin(2 \cdot Lat) \cdot \cos Az_M \cdot \sin H_{Mapp} - \sin^2 Lat \cdot \cos H_{Mapp}\right]$$

f is the flattening of the earth: $f = \frac{1}{298.257}$

$$P_{M,\,improved} = P_M + \Delta P_M$$

$$H_M = H_{Mapp} - R_{Mapp} + P_{M,\,improved}$$

Parallax in azimuth:

The correction for the parallax in azimuth is applied after calculating H_M and D. The following formula is a fairly accurate approximation of the parallax in azimuth, ΔAz_M:

$$\Delta Az_M \approx f \cdot HP_M \cdot \frac{\sin(2 \cdot Lat) \cdot \sin Az_M}{\cos H_M}$$

In order to find, how ΔAz_M affects D, we go back to the cosine formula:

$$\cos D = \sin H_M \cdot \sin H_B + \cos H_M \cdot \cos H_B \cdot \cos \alpha$$

We differentiate the equation with respect to α:

$$\frac{d(\cos D)}{d\alpha} = -\cos H_M \cdot \cos H_B \cdot \sin \alpha$$

$$d(\cos D) = -\cos H_M \cdot \cos H_B \cdot \sin \alpha \cdot d\alpha$$

$$d\left(\cos D\right) = -\sin D \cdot dD$$

$$-\sin D \cdot dD = -\cos H_M \cdot \cos H_B \cdot \sin \alpha \cdot d\alpha$$

$$dD = \frac{\cos H_M \cdot \cos H_B \cdot \sin \alpha}{\sin D} \cdot d\alpha$$

Since $d\alpha = d\,Az_M$, the change in D caused by an infinitesimal change in Az_M is:

$$dD = \frac{\cos H_M \cdot \cos H_B \cdot \sin \alpha}{\sin D} \cdot dAz_M$$

With a small but measurable change in Az_M, we have:

$$\Delta D \approx \frac{\cos H_M \cdot \cos H_B \cdot \sin \alpha}{\sin D} \cdot \Delta Az_M$$

$$D_{improved} \approx D + \Delta D$$

Combining the formulas for ΔAz_M and ΔD, we get:

$$D_{improved} \approx D + f \cdot HP_M \cdot \frac{\cos H_B \cdot \sin\left(2 \cdot Lat\right) \cdot \sin Az_M \cdot \sin\left(Az_M - Az_B\right)}{\sin D}$$

Accuracy

According to modern requirements, the lunar distance method is horribly inaccurate. In the 18th and early 19th century, however, this was generally accepted because a longitude with an error of 0.5°-1° was still better than no longitude at all. Said error is the approximate result of an error of only 1' in the measurement of D_{Lapp}, not uncommon for a sextant reading under practical conditions. Therefore, D_{Lapp} should be measured with greatest care.

The altitudes of both bodies do not quite require the same degree of precision because a small error in the apparent altitude leads to about the same error in the geocentric altitude. Since both errors cancel each other to a large extent, the resulting error in D is comparatively small. An altitude error of a few arcminutes is tolerable in most cases. Therefore, measuring two altitudes of each body and finding the altitude at the moment of the lunar distance observation by interpolation is not absolutely necessary. Measuring a single altitude of each body shortly before or after the lunar distance measurement is sufficient if a small loss in accuracy is accepted.

The position of the reference body with respect to the moon is crucial. The standard deviation of a time value obtained by lunar distance is inversely proportional to the rate of change of D. Since the plane of the lunar orbit forms a relatively small angle (approx. 5°) with the ecliptic, bright bodies in the vicinity of the ecliptic are most suitable (sun, planets, selected stars). The stars generally recommended for the lunar distance method are Aldebaran, Altair, Antares, Fomalhaut, Hamal, Markab, Pollux, Regulus, and Spica, but other stars close to the ecliptic may be used as well, e. g., Nunki. The lunar distance tables of the *Nautical Almanac* contained only D values for those bodies having a favorable position with respect to the moon on the day of observation. If in doubt, the navigator should check the rate of change of D. The latter becomes zero when D passes through a minimum or maximum, making an observation useless.