

Finding One's Position (Sight Reduction)

Lines of Position

Any geometrical or physical line passing through the observer's (still unknown) position and accessible through measurement or observation is called a **line of position** or **position line**, **LOP**. Examples are circles of equal altitude, meridians, parallels of latitude, bearing lines (compass bearings) of terrestrial objects, coastlines, rivers, roads, railroad tracks, power lines, etc. **A single position line indicates an infinite series of possible positions.** The observer's actual position is marked by the point of intersection of at least two position lines, regardless of their nature. A position thus found is called **fix** in navigator's language. The concept of the position line is essential to modern navigation.

Sight Reduction

Finding a line of position by observation of a celestial object is called **sight reduction**. Although some background in mathematics is required to comprehend the process completely, knowing the basic concepts and a few equations is sufficient for most practical applications. The geometrical background (law of cosines, navigational triangle) is given in chapter 10 and 11. In the following, we will discuss the semi-graphic methods developed by *Sumner* and *St. Hilaire*. Both methods require relatively simple calculations only and enable the navigator to plot lines of position on a navigation chart or **plotting sheet** (see chapter 13).

Knowing altitude and GP of a body, we also know the radius of the corresponding circle of equal altitude (our line of position) and the location of its center. As mentioned in chapter 1 already, plotting circles of equal altitude on a chart is usually impossible due to their large dimensions and the distortions caused by map projection. However, *Sumner* and *St. Hilaire* showed that only a short arc of each circle of equal altitude is needed to find one's position. Since this arc is comparatively short, it can be replaced with a secant or tangent a of the circle.

Local Meridian, Local Hour Angle and Meridian Angle

A meridian passing through a given position is called **local meridian**. In celestial navigation, the angle formed by the hour circle of the observed body (upper branch) and the local meridian (upper branch) plays a fundamental role. In analogy with the Greenwich hour angle, we can measure this angle westward from the local meridian (0°...+360°). In this case, the angle is called **local hour angle**, **LHA**. It is also possible to measure the angle westward (0°...+180°) or eastward (0°...−180°) from the local meridian in wich case it is called **meridian angle**, **t**. In most navigational formulas, LHA and t can be substituted for each other since the trigonometric functions return the same results for either of both angles. For example, the cosine of +315° is the same as the cosine of −45°.

LHA as well as t is the algebraic sum of the Greenwich hour angle of the body, GHA, and the observer's longitude, Lon. To make sure that the obtained angle is in the desired range, the following rules have to be applied when forming the sum of GHA and Lon:

$$LHA = \begin{cases} GHA + Lon & \text{if} \quad 0° < GHA + Lon < 360° \\ GHA + Lon + 360° & \text{if} \quad GHA + Lon < 0° \\ GHA + Lon - 360° & \text{if} \quad GHA + Lon > 360° \end{cases}$$

$$t = \begin{cases} GHA + Lon & \text{if} \quad GHA + Lon < 180° \\ GHA + Lon - 360° & \text{if} \quad GHA + Lon > 180° \end{cases}$$

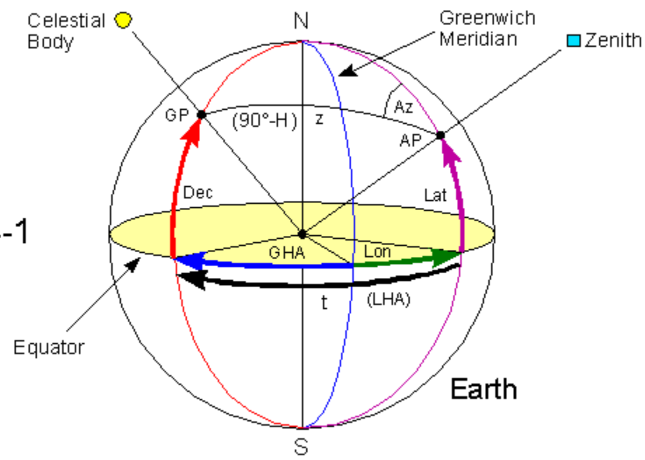
In all calculations, the sign of Lon and t, respectively, has to be observed carefully. The sign convention is:

Eastern longitude:	**positive**
Western longitude:	**negative**
Eastern meridian angle:	**negative**
Western meridian angle:	**positive**

For reasons of symmetry, we will refer to the meridian angle in the following considerations although the local hour angle would lead to the same results (a body has the same altitude with the meridian angle +t and −t, respectively).

Fig. 4-1 illustrates the various angles involved in the sight reduction process.

Fig. 4-1

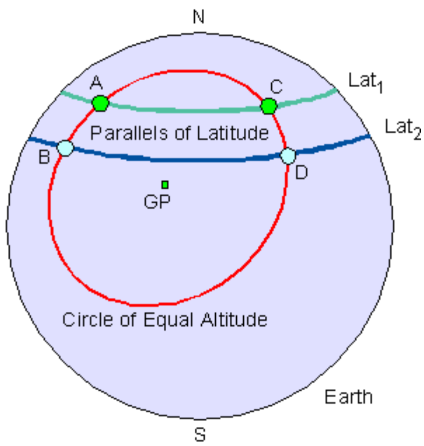


Sumner's Method

In December 1837, *Thomas H Sumner*, an American sea captain, was on a voyage from South Carolina to Greenock, Scotland. When approaching *St. George's Channel* between Ireland and Wales, he managed to measure a single altitude of the sun after a longer period of bad weather. Using the time sight formula (see chapter 6), he calculated a longitude from his estimated latitude. Since he was doubtful about his estimate, he repeated his calculations with two slightly different latitudes. To his surprise, the three points thus obtained were on a straight line. Accidentally, the line passed through the position of a light house off the coast of Wales (*Small's Light*). By intuition, Sumner steered his ship along this line and soon after, *Small's Light* came in sight. Sumner concluded that he had found a "line of equal altitude". The publication of his method in 1843 marked the beginning of "modern" celestial navigation [18]. Although rarely used today, it is still an interesting alternative. It is easy to comprehend and the calculations to be done are extremely simple.

Fig. 4-2 illustrates the points where a circle of equal altitude intersects two chosen parallals of latitude.

Fig. 4-2



An observer being between Lat_1 and Lat_2 is either on the arc A-B or on the arc C-D. With a rough estimate of his longitude, the observer can easily find on which of both arcs he is, for example, A-B. The arc thus found is the relevant part of his line of position, the other arc is discarded. We can approximate the line of position by drawing a straight line through A and B which is a secant of the circle of equal altitude. This secant is called **Sumner line**. Before plotting the Sumner line on our chart, we have to find the longitude of each point of intersection, A, B, C, and D.

Procedure:

1.

We choose a parallel of latitude (Lat_1) north of our estimated latitude. Preferably, Lat_1 should be marked by the nearest horizontal grid line on our chart or plotting sheet.

2.

From Lat_1 , Dec, and the observed altitude, Ho, we calculate the meridian angle, t, using the following formula:

$$t = \pm \arccos \frac{\sin Ho - \sin Lat \cdot \sin Dec}{\cos Lat \cdot \cos Dec}$$

The equation is derived from the navigational triangle (chapter 10 & 11). It has two solutions, +t and –t, since the cosine of +t equals the cosine of –t. Geometrically, this corresponds with the fact that the circle of equal altitude intersects the parallel of latitude at two points. Using the following formulas and rules, we obtain the longitudes of these points of intersection, Lon and Lon':

$$Lon = t - GHA$$

$$Lon' = 360° - t - GHA$$

$$\begin{array}{lll} \text{If } Lon < -180° & \rightarrow & Lon + 360° \\ \text{If } Lon' < -180° & \rightarrow & Lon' + 360° \\ \text{If } Lon' > +180° & \rightarrow & Lon' - 360° \end{array}$$

Comparing the longitudes thus obtained with our estimate, we select the relevant longitude and discard the other one. This method of finding one's longitude is called **time sight** (see chapter 6).

3.

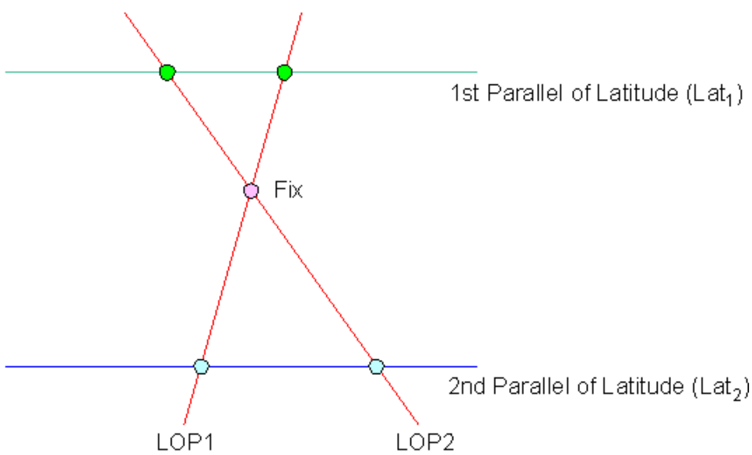
We chose a parallel of latitude (Lat_2) south of our estimated latitude. The difference between Lat_1 and Lat_2 should not exceed 1 or 2 degrees. We repeat steps 1 and 2 with the second latitude, Lat_2.

4.

On our plotting sheet, we mark each remaining longitude on the corresponding parallel and plot the Sumner line through the points thus located (LOP1).

To obtain a fix, we repeat steps 1 through 4 with the same parallels and the declination and observed altitude of a second body. The point where the Sumner line thus obtained, LOP2, intersects LOP1 is our fix (*Fig. 4-3*).

Fig. 4-3



If we have only a very rough estimate of our latitude, the point of intersection may be slightly outside the interval defined by both parallels. Nevertheless, the fix is correct. A fix obtained with *Sumner's* method has a small error caused by neglecting the curvature of the circles of equal altitude. We can improve the fix by iteration. In this case, we choose a new pair of assumed latitudes, nearer to the fix, and repeat the procedure. Ideally, the horizontal distance between both bodies should be 90° (30°...150° is tolerable). Otherwise, the fix would become indistinct. Further, neither of the bodies should be near the local meridian (see time sight, chapter 6). *Sumner's* method has the (small) advantage that no protractor is needed to plot lines of position.

The Intercept Method

This procedure was developed by the French navy officer *St. Hilaire* and others and was first published in 1875. After that, it gradually became the standard for sight reduction since it avoids some of the restrictions of *Sumner's* method. Although the background is more complicated than with *Sumner's* method, the practical application is very convenient.

Theory:

For any given position of the observer, the altitude of a celestial body, reduced to the celestial horizon, is **solely** a function of the observer's latitude, the declination of the body, and the meridian angle (or local hour angle). The **altitude formula** is obtained by applying the law of cosine for sides to the navigational triangle (see chapter 10 & 11):

$$H = \arcsin\left(\sin Lat \cdot \sin Dec + \cos Lat \cdot \cos Dec \cdot \cos t\right)$$

We choose an arbitrary point in the vicinity of our estimated position, preferably the nearest point where two grid lines on the chart intersect. This point is called **assumed position**, **AP** (*Fig. 4-2*). Using the above formula, we calculate the altitude of the body resulting from Lat_{AP} and Lon_{AP}, the geographic coordinates of AP. The altitude thus obtained is called **computed** or **calculated altitude**, **Hc**.

Usually, Hc will slightly differ from the actually **observed altitude**, **Ho** (see chapter 2). The difference, ΔH, is called **intercept**.

$$\Delta H = Ho - Hc$$

Ideally, Ho and Hc are identical if the observer is at AP.

In the following, we will discuss which possible positions of the observer would result in the same intercept, ΔH. For this purpose, we assume that the intercept is an infinitesimal quantity and denote it by dH. The general formula is:

$$dH = \frac{\partial H}{\partial Lat} \cdot dLat + \frac{\partial H}{\partial t} \cdot dt$$

This differential equation has an infinite number of solutions. Since dH and both differential coefficients are constant, it can be reduced to an equation of the general form:

$$dLat = a + b \cdot dt$$

Thus, the graph is a straight line, and it is sufficient to dicuss two special cases, dt=0 and dLat=0, respectively.

In the first case, the observer is on the same meridian as AP, and dH is solely caused by a small change in latitude, dLat, whereas t is constant (dt = 0). We differentiate the altitude formula with respect to Lat:

$$\sin H = \sin Lat \cdot \sin Dec + \cos Lat \cdot \cos Dec \cdot \cos t$$

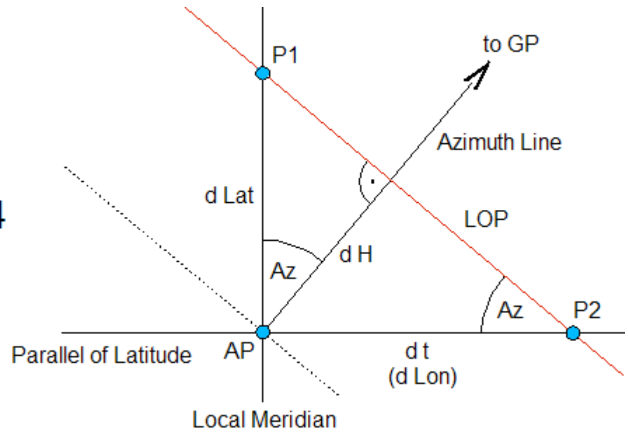
$$d\left(\sin H\right) = \left(\cos Lat \cdot \sin Dec - \sin Lat \cdot \cos Dec \cdot \cos t\right) \cdot dLat$$

$$\cos H \cdot dH = \left(\cos Lat \cdot \sin Dec - \sin Lat \cdot \cos Dec \cdot \cos t\right) \cdot dLat$$

$$dLat = \frac{\cos H}{\cos Lat \cdot \sin Dec - \sin Lat \cdot \cos Dec \cdot \cos t} \cdot dH$$

Adding dLat to Lat_{AP}, we obtain the point P1, as illustrated in *Fig.4-4*. P1 is on the observer's circle of equal altitude.

Fig. 4-4



In the second case, the observer is on the same parallel of latitude as AP, and dH is solely caused by a small change in the meridian angle, dt, whereas Lat is constant (dLat=0). We differentiate the altitude formula with respect to t:

$$\sin H = \sin Lat \cdot \sin Dec + \cos Lat \cdot \cos Dec \cdot \cos t$$

$$d\left(\sin H\right) = -\cos Lat \cdot \cos Dec \cdot \sin t \cdot dt$$

$$\cos H \cdot dH = -\cos Lat \cdot \cos Dec \cdot \sin t \cdot dt$$

$$dt = -\frac{\cos H}{\cos Lat \cdot \cos Dec \cdot \sin t} \cdot dH$$

Adding dt (corresponding with an equal change in longitude, dLon) to Lon_{AP}, we obtain the point P2 which, too, is on the observer's circle of equal altitude. Thus, we would measure Ho at P1 and P2, respectively. Knowing P1 and P2, we can plot a straight line passing through these positions. This line is a tangent of the circle of equal altitude and is our **line of position**, **LOP**. The great circle passing through AP and GP is represented by a straight line perpendicular to the line of position. The arc between AP and GP is the radius of the circle of equal altitude. The distance between AP and the point where this line, called **azimuth line**, intersects the line of position is the intercept, dH. The angle formed by the azimuth line and the local meridian of AP is called **azimuth angle**, **Az**. The same angle is formed by the line of position and the parallel of latitude passing through AP (*Fig. 4-4*).

There are several ways to obtain Az and the true azimuth, Az_N, from the right (plane) triangle formed by AP, P1, and P2:

1. Time-altitude azimuth:

$$\cos Az = \frac{dH}{dLat} = \frac{\cos Lat \cdot \sin Dec - \sin Lat \cdot \cos Dec \cdot \cos t}{\cos H}$$

or

$$Az = \arccos\left(\frac{\cos Lat \cdot \sin Dec - \sin Lat \cdot \cos Dec \cdot \cos t}{\cos H}\right)$$

Az is not necessarily identical with the true azimuth, Az_N, since the arccos function returns angles between 0° and +180°, whereas Az_N is measured from 0° to +360°.

To obtain Az_N, we have to apply the following rules when using the formula for time-altitude azimuth:

$$Az_N = \begin{cases} Az & \text{if} \quad t < 0° \quad (\text{or } 180° < LHA < 360°) \\ 360° - Az & \text{if} \quad t > 0° \quad (\text{or } 0° < LHA < 180°) \end{cases}$$

2. Time azimuth:

$$\tan Az = \frac{d\,Lat}{\cos Lat \cdot dt} = \frac{\sin t}{\sin Lat \cdot \cos t - \cos Lat \cdot \tan Dec}$$

The factor cos Lat is the relative circumference of the parallel of latitude going through AP (equator = 1).

$$Az = \arctan \frac{\sin t}{\sin Lat \cdot \cos t - \cos Lat \cdot \tan Dec}$$

The time azimuth formula does not require the altitude. Since the arctan function returns angles between -90° and +90°, a different set of rules is required to obtain Az_N:

$$Az_N = \begin{cases} Az & \text{if} \quad \text{numerator} < 0 \quad \text{AND} \quad \text{denominator} < 0 \\ Az + 360° & \text{if} \quad \text{numerator} > 0 \quad \text{AND} \quad \text{denominator} < 0 \\ Az + 180° & \text{if} \quad \text{denominator} > 0 \end{cases}$$

3. Alternative formula:

$$\sin Az = \frac{dh}{\cos Lat \cdot dt} = -\frac{\cos Dec \cdot \sin t}{\cos h}$$

or

$$Az = \arcsin\left(-\frac{\cos Dec \cdot \sin t}{\cos h}\right)$$

Interestingly, this formula does not require the latitude. The accompanying rules for Az_N are:

$$Az_N = \begin{cases} Az & \text{if} \quad \text{Dec} > 0 \quad \text{AND} \quad \text{t} < 0 \\ 360° + Az & \text{if} \quad \text{Dec} > 0 \quad \text{AND} \quad \text{t} > 0 \\ 180° - Az & \text{if} \quad \text{Dec} < 0 \end{cases}$$

4. Altitude azimuth:

This formula is directly derived from the navigational triangle (cosine law, see chapter 10 & 11) without using differential calculus.

$$\cos Az = \frac{\sin Dec - \sin H \cdot \sin Lat}{\cos H \cdot \cos Lat}$$

or

$$Az = \arccos \frac{\sin Dec - \sin Hc \cdot \sin Lat}{\cos Hc \cdot \cos Lat}$$

As with the formula for time-altitude azimuth, Az_N is obtained through these rules:

$$Az_N = \begin{cases} Az & \text{if} \quad t < 0° \quad (\text{or } 180° < LHA < 360°) \\ 360° - Az & \text{if} \quad t > 0° \quad (\text{or } 0° < LHA < 180°) \end{cases}$$

In contrast to dH, ΔH is a measurable quantity, and the position line is curved. *Fig. 4-5* shows a macroscopic view of the line of position, the azimuth line, and the circles of equal altitude.

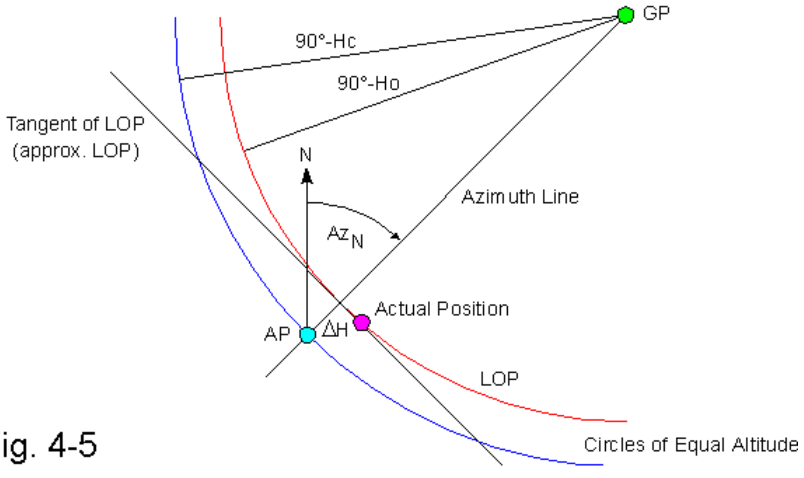


Fig. 4-5

Procedure

Although the theory of the intercept method looks complicated, its practical application is very simple and does not require any background in differential calculus. The procedure comprises the following steps:

1.

We choose an **assumed position**, **AP**, near to our **estimated position**. Preferably, AP should be defined by an integer number of degrees for Lat_{AP} and Lon_{AP}, respectively, depending on the scale of the chart. Instead of AP, our estimated position itself may be used. Plotting lines of position, however, is more convenient when putting AP on the point of intersection of two grid lines.

2.

We calculate the meridian angle, t_{AP}, (or local hour angle, LHA_{AP}) from GHA and Lon_{AP}, as stated above.

3.

We calculate the altitude of the observed body as a function of Lat_{AP}, t_{AP}, and Dec (computed altitude):

$$Hc = \arcsin\left(\sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos t_{AP}\right)$$

4.

Using one of the azimuth formulas stated above, we calculate the **true azimuth** of the body, $\mathbf{Az_N}$, from Hc, Lat_{AP}, t_{AP}, and Dec, for example:

$$Az = \arccos \frac{\sin Dec - \sin Hc \cdot \sin Lat_{AP}}{\cos Hc \cdot \cos Lat_{AP}}$$

$$Az_N = \begin{cases} Az & \text{if} \quad t < 0° \quad (\text{or } 180° < LHA < 360°) \\ 360° - Az & \text{if} \quad t > 0° \quad (\text{or } 0° < LHA < 180°) \end{cases}$$

5.

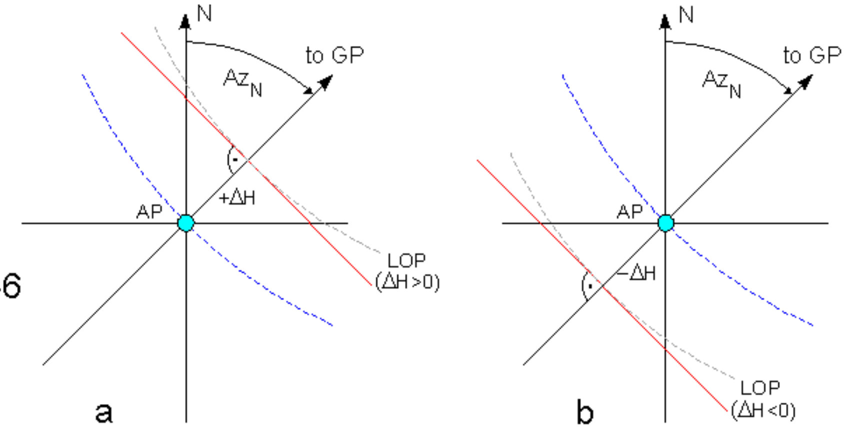
We calculate the **intercept**, **ΔH**, the difference between observed altitude, Ho (chapter 2), and computed altitude, Hc. The intercept, which is directly proportional to the difference between the radii of the corresponding circles of equal altitude, is usually expressed in nautical miles:

$$\Delta H\left[nm\right] = 60 \cdot \left(Ho\left[°\right] - Hc\left[°\right] \right)$$

6.

On the chart, we draw a suitable part of the azimuth line through AP (*Fig. 4-6*). On this line, we measure the intercept, ΔH, from AP (towards GP if ΔH>0, away from GP if ΔH<0) and draw a perpendicular through the point thus located. This perpendicular is our approximate line of position (red line).

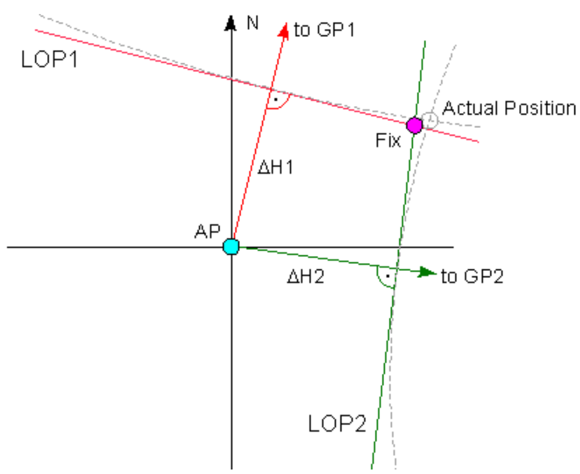
Fig. 4-6



7.

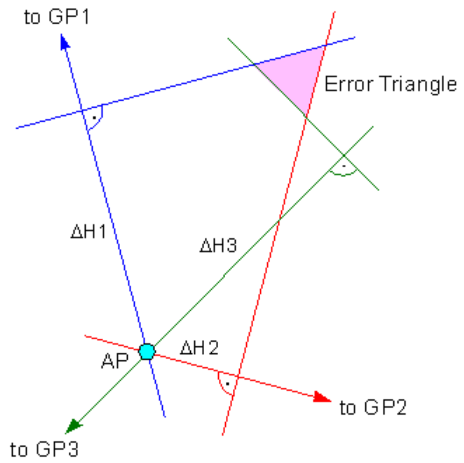
To obtain our position, we need at least one more line of position. We repeat the procedure with altitude and GP of a second celestial body or of the same body at a different time of observation (*Fig. 4-4*). The point where both position lines (tangents) intersect is our fix. The second observation does not necessarily require the same AP to be used.

Fig. 4-7



As mentioned above, the intercept method ignores the curvature of the actual LoP's. Therefore, the obtained fix is not our exact position but an **improved position** (compared with AP). The residual error remains tolerable as long as the radii of the circles of equal altitude are great enough and AP is not too far from the actual position (see chapter 16). The geometric error inherent to the intercept method can be decreased by **iteration**, i.e., substituting the obtained fix for AP and repeating the calculations (same altitudes and GP's). This will result in a more accurate position. If necessary, we can reiterate the procedure until the obtained position remains virtually constant. Since an estimated position is usually nearer to our true position than an assumed position, the latter may require a greater number of iterations. Accuracy is also improved by observing three bodies instead of two. Theoretically, the position lines should intersect each other at a single point. Since no observation is entirely free of errors, we will usually obtain three points of intersection forming an **error triangle** (*Fig. 4-8*).

Fig. 4-8



Area and shape of the triangle give us a rough estimate of the quality of our observations (see chapter 16). Our **most probable position**, **MPP**, is approximately at the center of the inscribed circle of the error triangle (the point where the bisectors of the three angles of the error triangle meet).

When observing more than three bodies, the resulting position lines will form the corresponding polygons.

Direct Computation

If we do not want to plot lines of position to determine our fix, we can calculate the most probable position directly from an unlimited number of observations, n (n > 1). The Nautical Almanac provides an averaging procedure. First, the auxiliary quantities A, B, C, D, E, and G have to be calculated:

$$A = \sum_{i=1}^{n} \cos^2 Az_i \qquad B = \sum_{i=1}^{n} \sin Az_i \cdot \cos Az_i \qquad C = \sum_{i=1}^{n} \sin^2 Az_i$$

$$D = \sum_{i=1}^{n} (\Delta H)_i \cdot \cos Az_i \qquad E = \sum_{i=1}^{n} (\Delta H)_i \cdot \sin Az_i \qquad G = A \cdot C - B^2$$

In these formulas, Az_i denotes the *true* azimuth of the respective body. The ΔH values are measured in degrees (same unit as Lon and Lat). The geographic coordinates of the observer's MPP are then obtained as follows:

$$Lon = Lon_{AP} + \frac{A \cdot E - B \cdot D}{G \cdot \cos Lat_{AP}} \qquad Lat = Lat_{AP} + \frac{C \cdot D - B \cdot E}{G}$$

The method does not correct for the geometric errors inherent to the intercept method. These are eliminated, if necessary, by iteration. For this purpose, we substitute the calculated MPP for AP. For each body, we calculate new values for t (or LHA), Hc, ΔH, and Az_N. With these values, we recalculate A, B, C, D, E, G, Lon, and Lat.

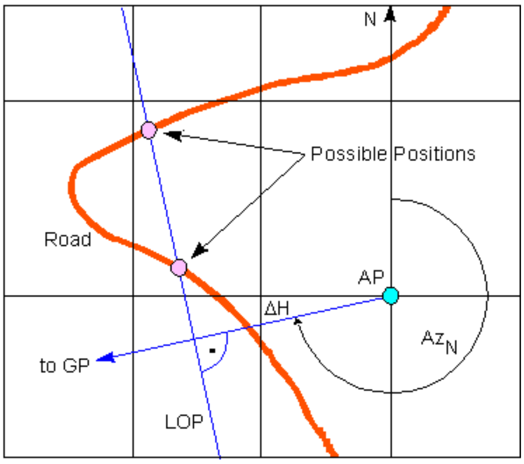
Repeating this procedure, the resulting positions will converge rapidly. In the majority of cases, one or two iterations will be sufficient, depending on the distance between AP and the true position.

Combining Different Lines of Position

Since the point of intersection of any two position lines, regardless of their nature, marks the observer's geographic position, one celestial LOP may suffice to find one's position if another LOP of a different kind is available.

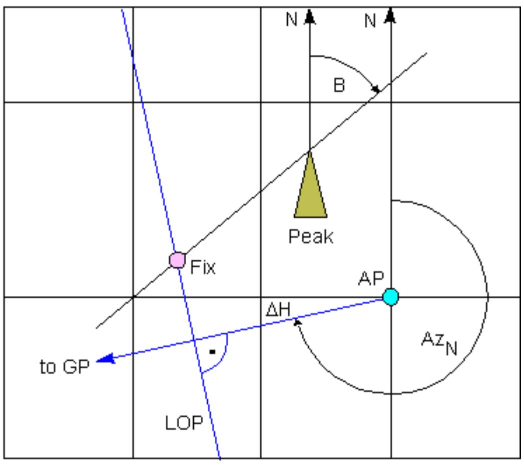
In the desert, for instance, we can determine our current position by finding the point on the map where a position line obtained by observation of a celestial object intersects the dirt road we are using (*Fig. 4-9*).

Fig. 4-9



We can as well find our position by combining our celestial LOP with the bearing line of a distant mountain peak or any other prominent landmark (*Fig. 4-10*). B is the compass bearing of the terrestrial object (corrected for magnetic declination).

Fig. 4-10



Both examples demonstrate the versatility of position line navigation.