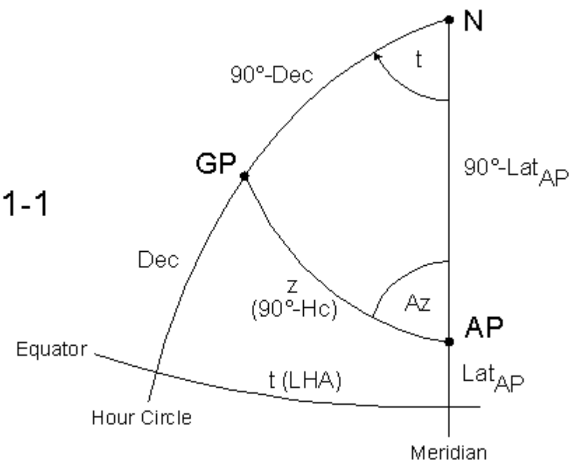


The Navigational Triangle

The **navigational triangle** is the (usually oblique) spherical triangle on the earth's surface formed by the north pole, N, the observer's assumed position, AP, and the geographic position of the celestial object, GP, at the time of observation (*Fig. 11-1*). **All common sight reduction procedures are based upon the navigational triangle.**

Fig. 11-1



When using the intercept method, the latitude of the assumed position, Lat_{AP}, the declination of the observed celestial body, Dec, and the meridian angle, t, or the local hour angle, LHA, (calculated from the longitude of AP and the GHA of the object), are initially known to the observer.

The first step is calculating the side z of the navigational triangle by using the law of cosines for sides:

$$\cos z = \cos\left(90° - Lat_{AP}\right) \cdot \cos\left(90° - Dec\right) + \sin\left(90° - Lat_{AP}\right) \cdot \sin\left(90° - Dec\right) \cdot \cos t$$

Since cos (90°-x) equals sin x and vice versa, the equation can be written in a simpler form:

$$\cos z = \sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos t$$

The side z is not only the **great circle distance** between AP and GP but also the **zenith distance** of the celestial object and the radius of the **circle of equal altitude** (see chapter 1).

Substituting the altitude H for z, we get:

$$\sin H = \sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos t$$

Solving the equation for H leads to the altitude formula known from chapter 4:

$$H = \arcsin\left(\sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos t\right)$$

The altitude thus obtained for a given position is called **computed altitude**, **Hc**.

The azimuth angle of the observed body is also calculated by means of the law of cosines for sides:

$$\cos\left(90° - Dec\right) = \cos\left(90° - Lat_{AP}\right) \cdot \cos z + \sin\left(90° - Lat_{AP}\right) \cdot \sin z \cdot \cos Az$$

$$\sin Dec = \sin Lat_{AP} \cdot \cos z + \cos Lat_{AP} \cdot \sin z \cdot \cos Az$$

Using the computed altitude instead of the zenith distance results in the following equation:

$$\sin Dec = \sin Lat_{AP} \cdot \sin Hc + \cos Lat_{AP} \cdot \cos Hc \cdot \cos Az$$

Solving the equation for Az finally yields the azimuth formula from chapter 4:

$$Az = \arccos \frac{\sin Dec - \sin Lat_{AP} \cdot \sin Hc}{\cos Lat_{AP} \cdot \cos Hc}$$

The resulting azimuth angle is always in the range of 0°... 180° and therefore not necessarily identical with the true azimuth, Az_N (0°... 360° clockwise from true north) commonly used in navigation. In all cases where t is negative (GP east of AP) , Az equals Az_N. Otherwise (t positive, GP westward from AP as shown in *Fig. 11-1*), Az_N is obtained by subtracting Az from 360°.

When using *Sumner's* method, Dec, and Lat_{AP} (the assumed latitude) are the known quantities, and z (or H) is measured. The meridian angle, t, (or the local hour angle, LHA) is the quantity to be calculated.

Again, the law of cosines for sides is applied:

$$\cos z = \cos\left(90° - Lat_{AP}\right) \cdot \cos\left(90° - Dec\right) + \sin\left(90° - Lat_{AP}\right) \cdot \sin\left(90° - Dec\right) \cdot \cos t$$

$$\sin H = \sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos t$$

$$\cos t = \frac{\sin H - \sin Lat_{AP} \cdot \sin Dec}{\cos Lat_{AP} \cdot \cos Dec}$$

$$t = \arccos \frac{\sin H - \sin Lat_{AP} \cdot \sin Dec}{\cos Lat_{AP} \cdot \cos Dec}$$

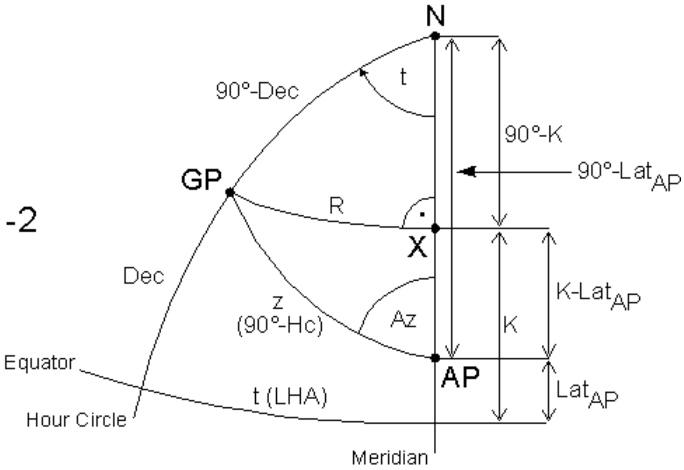
The obtained meridian angle, t, (or LHA) is then used as described in chapter 4.

When observing a celestial body at the time of meridian passage, the local hour angle is zero, and the navigational triangle becomes infinitesimally narrow. In this special case, the formulas of spherical trigonometry are not needed, and the sides of the spherical triangle can be calculated by simple addition or subtraction.

The Divided Navigational Triangle

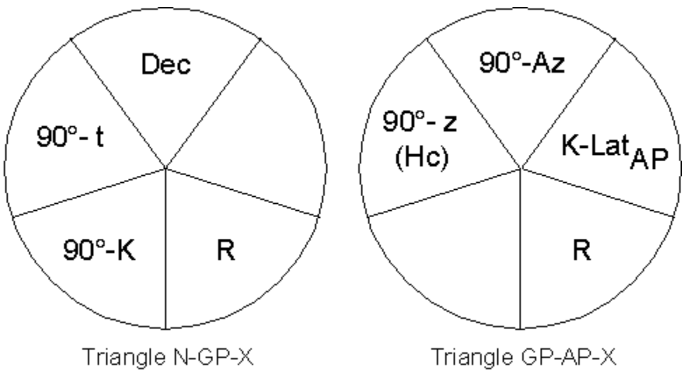
An alternative method for solving the navigational triangle is based upon two right spherical triangles obtained by constructing a great circle passing through GP and intersecting the local meridian perpendicularly at X (*Fig. 11-2*):

Fig. 11-2



The first right triangle is formed by GP, N, and X, the second one by GP, X, and AP. The **auxiliary parts** R and K are intermediate quantities used to calculate z (or Hc) and Az. K is the angular distance of X from the equator, measured through AP. Both triangles are solved using **Napier's Rules of Circular Parts** (see chapter 9). *Fig. 11-3* illustrates the corresponding circular diagrams:

Fig. 11-3



According to Napier's rules, Hc and Az are calculated by means of the following formulas:

$$\sin R = \sin t \cdot \cos Dec \qquad \Rightarrow \qquad R = \arcsin\left(\sin t \cdot \cos Dec\right)$$

$$\sin Dec = \cos R \cdot \sin K \qquad \Rightarrow \qquad \sin K = \frac{\sin Dec}{\cos R} \qquad \Rightarrow \qquad K = \arcsin \frac{\sin Dec}{\cos R}$$

Substitute 180°–K for K in the following equation if $|t| > 90°$ (or $90° < LHA < 270°$).

$$\sin Hc = \cos R \cdot \cos\left(K - Lat_{AP}\right) \qquad \Rightarrow \qquad Hc = \arcsin\left[\cos R \cdot \cos\left(K - Lat_{AP}\right)\right]$$

$$\sin R = \cos Hc \cdot \sin Az \qquad \Rightarrow \qquad \sin Az = \frac{\sin R}{\cos Hc} \qquad \Rightarrow \qquad Az = \arcsin \frac{\sin R}{\cos Hc}$$

For further calculations, substitute 180°–Az for Az if K and Lat have opposite signs or if $|K| < |Lat|$.

To obtain the true azimuth, Az_N (0°... 360°), the following rules have to be applied:

$$Az_N = \begin{cases} -Az & \text{if} \quad Lat_{AP} > 0\,(\text{N}) \quad \text{AND} \quad t < 0 \ \left(180° < LHA < 360°\right) \\ 360° - Az & \text{if} \quad Lat_{AP} > 0\,(\text{N}) \quad \text{AND} \quad t > 0 \ \left(0° < LHA < 180°\right) \\ 180° + Az & \text{if} \quad Lat_{AP} < 0\,(\text{S}) \end{cases}$$

The divided navigational triangle is of considerable importance since it forms the theoretical background for a number of **sight reduction tables**, e.g., the Ageton Tables.

Using the secant and cosecant function (sec x = 1/cos x, csc x = 1/sin x), the equations for the divided navigational triangle are stated as:

$$\csc R = \csc t \cdot \sec Dec$$

$$\csc K = \frac{\csc Dec}{\sec R}$$

Substitute 180°–K for K in the following equation if $|t| > 90°$.

$$\csc Hc = \sec R \cdot \sec\left(K - Lat\right)$$

$$\csc Az = \frac{\csc R}{\sec Hc}$$

Substitute 180°–Az for Az if K and Lat have opposite signs or if $|K| < |Lat|$.

In logarithmic form, these equations are stated as:

$$\log\csc R = \log\csc t + \log\sec Dec$$

$$\log\csc K = \log\csc Dec - \log\sec R$$

$$\log\csc Hc = \log\sec R + \log\sec\left(K - Lat\right)$$

$$\log\csc Az = \log\csc R - \log\sec Hc$$

Having the logarithms of the secants and cosecants of angles available in the form of a suitable table, we can solve a sight by a sequence of simple additions and subtractions (beside converting the angles to their corresponding log secants and log cosecants and vice versa). Apart from the table itself, the only tools required are a sheet of paper and a pencil.

The Ageton Tables (H.O. 211), first published in 1931, are based upon the above formulas and provide a very efficient arrangement of angles and their log secants and log cosecants on 36 pages. Since all calculations are based on absolute values, certain rules included in the instructions have to be observed.

Sight reduction tables were developed many years before electronic calculators became available in order to simplify calculations necessary to reduce a sight. Still today, sight reduction tables are preferred by people who do not want to deal with the formulas of spherical trigonometry. Moreover, they provide a valuable backup method if electronic devices fail.

Two modified versions of the Ageton Tables are available at: http://www.celnav.de/page3.htm