

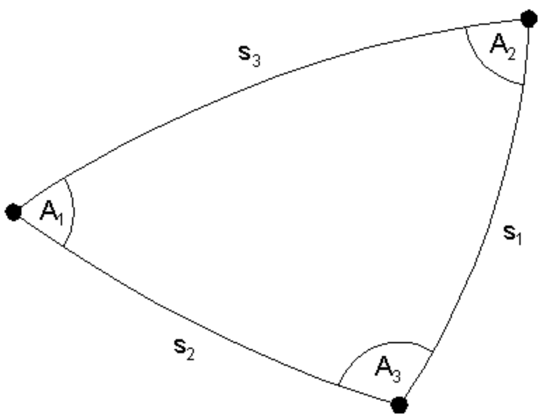
Spherical Trigonometry

The earth is usually regarded as a sphere in celestial navigation although an oblate spheroid would be a better approximation. Otherwise, navigational calculations would become too difficult for practical use. The position error introduced by the spherical earth model is usually very small and stays within the "statistical noise" caused by other omnipresent errors like, e.g., abnormal refraction, rounding errors, etc. Although it is possible to perform navigational calculations solely with the aid of tables (H.O. 229, H.O. 211, etc.) and with little mathematics, the principles of celestial navigation can not be comprehended without knowing the elements of spherical trigonometry.

The Oblique Spherical Triangle

Like any triangle, a spherical triangle is characterized by three sides and three angles. However, a spherical triangle is part of the surface of a sphere, and the sides are not straight lines but arcs of great circles (*Fig. 10-1*).

Fig. 10-1



A **great circle** is a circle on the surface of a sphere whose plane passes through the center of the sphere (see chapter 3).

Any side of a spherical triangle can be regarded as an angle - the angular distance between the adjacent vertices, measured at the center of the sphere. The interrelations between angles and sides of a spherical triangle are described by the **law of sines**, the **law of cosines for sides**, the **law of cosines for angles**, **Napier's analogies**, and **Gauss' formulas** (apart from other formulas).

Law of sines:

$$\frac{\sin A_1}{\sin s_1} = \frac{\sin A_2}{\sin s_2} = \frac{\sin A_3}{\sin s_3}$$

Law of cosines for sides:

$$\cos s_1 = \cos s_2 \cdot \cos s_3 + \sin s_2 \cdot \sin s_3 \cdot \cos A_1$$
$$\cos s_2 = \cos s_1 \cdot \cos s_3 + \sin s_1 \cdot \sin s_3 \cdot \cos A_2$$
$$\cos s_3 = \cos s_1 \cdot \cos s_2 + \sin s_1 \cdot \sin s_2 \cdot \cos A_3$$

Law of cosines for angles:

$$\cos A_1 = -\cos A_2 \cdot \cos A_3 + \sin A_2 \cdot \sin A_3 \cdot \cos s_1$$
$$\cos A_2 = -\cos A_1 \cdot \cos A_3 + \sin A_1 \cdot \sin A_3 \cdot \cos s_2$$
$$\cos A_3 = -\cos A_1 \cdot \cos A_2 + \sin A_1 \cdot \sin A_2 \cdot \cos s_3$$

Napier's analogies:

$$\tan\frac{A_1+A_2}{2}\cdot\tan\frac{A_3}{2} = \frac{\cos\frac{s_1-s_2}{2}}{\cos\frac{s_1+s_2}{2}} \qquad \tan\frac{A_1-A_2}{2}\cdot\tan\frac{A_3}{2} = \frac{\sin\frac{s_1-s_2}{2}}{\sin\frac{s_1+s_2}{2}}$$

$$\frac{\tan\frac{s_1+s_2}{2}}{\tan\frac{s_3}{2}} = \frac{\cos\frac{A_1-A_2}{2}}{\cos\frac{A_1+A_2}{2}} \qquad \frac{\tan\frac{s_1-s_2}{2}}{\tan\frac{s_3}{2}} = \frac{\sin\frac{A_1-A_2}{2}}{\sin\frac{A_1+A_2}{2}}$$

Gauss' formulas:

$$\frac{\sin\frac{A_1+A_2}{2}}{\cos\frac{A_3}{2}} = \frac{\cos\frac{s_1-s_2}{2}}{\cos\frac{s_3}{2}} \qquad \frac{\cos\frac{A_1+A_2}{2}}{\sin\frac{A_3}{2}} = \frac{\cos\frac{s_1+s_2}{2}}{\cos\frac{s_3}{2}}$$

$$\frac{\sin\frac{A_1-A_2}{2}}{\cos\frac{A_3}{2}} = \frac{\sin\frac{s_1-s_2}{2}}{\sin\frac{s_3}{2}} \qquad \frac{\cos\frac{A_1-A_2}{2}}{\sin\frac{A_3}{2}} = \frac{\sin\frac{s_1+s_2}{2}}{\sin\frac{s_3}{2}}$$

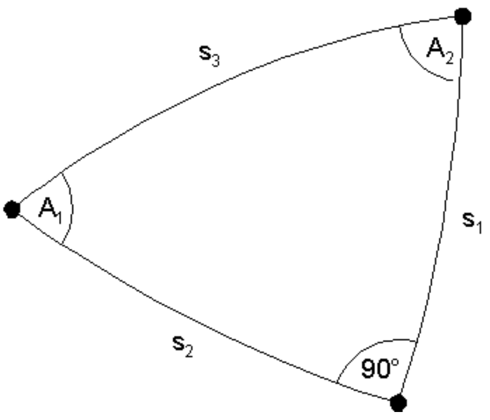
These formulas and others derived thereof enable any quantity (angle or side) of a spherical triangle to be calculated if three other quantities are known.

Particularly the law of cosines for sides is of interest to the navigator.

The Right Spherical Triangle

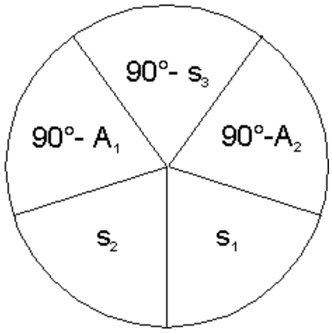
Solving a spherical triangle is less complicated when it contains a right angle (*Fig. 10-2*). Using ***Napier's* rules of circular parts**, any quantity can be calculated if only two other quantities (apart from the right angle) are known.

Fig. 10-2



We arrange the sides forming the right angle (s_1, s_2) and the **complements** of the remaining angles (A_1, A_2) and opposite side (s_3) in the form of a circular diagram consisting of five sectors, called "parts" (in the same order as they occur in the triangle). The right angle itself is omitted (*Fig. 10-3*):

Fig. 10-3



According to *Napier's* rules, the sine of any part of the diagram equals the product of the tangents of the adjacent parts and the product of the cosines of the opposite parts:

$$\sin s_1 = \tan s_2 \cdot \tan(90° - A_2) = \cos(90° - A_1) \cdot \cos(90° - s_3)$$
$$\sin s_2 = \tan(90° - A_1) \cdot \tan s_1 = \cos(90° - s_3) \cdot \cos(90° - A_2)$$
$$\sin(90° - A_1) = \tan(90° - s_3) \cdot \tan s_2 = \cos(90° - A_2) \cdot \cos s_1$$
$$\sin(90° - s_3) = \tan(90° - A_2) \cdot \tan(90° - A_1) = \cos s_1 \cdot \cos s_2$$
$$\sin(90° - A_2) = \tan s_1 \cdot \tan(90° - s_3) = \cos s_2 \cdot \cos(90° - A_1)$$

In a simpler form, these equations are stated as:

$$\sin s_1 = \tan s_2 \cdot \cot A_2 = \sin A_1 \cdot \sin s_3$$
$$\sin s_2 = \cot A_1 \cdot \tan s_1 = \sin s_3 \cdot \sin A_2$$
$$\cos A_1 = \cot s_3 \cdot \tan s_2 = \sin A_2 \cdot \cos s_1$$
$$\cos s_3 = \cot A_2 \cdot \cot A_1 = \cos s_1 \cdot \cos s_2$$
$$\cos A_2 = \tan s_1 \cdot \cot s_3 = \cos s_2 \cdot \sin A_1$$

Ageton's sight reduction tables, for example, are based upon the formulas of the right spherical triangle (chapter 11).